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Coincidence points of fiber maps on S^n-bundles

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ABSTRACT

In this note we study coincidence of pairs of fiber-preserving maps $f, g : E_1 \to E_2$ where E_1, E_2 are S^n-bundles over a space B. We will show that for each homotopy class $[f]$ of fiber-preserving maps over B, there is only one homotopy class $[g]$ such that the pair (f, g), where $[g] = [\tau \circ f]$ can be deformed to a coincidence free pair. Here $\tau : E_2 \to E_2$ is a fiber-preserving map which is fixed point free. In the case where the base is S^1 we classify the bundles, the homotopy classes of maps over S^1 and the pairs which can be deformed to coincidence free. At the end we discuss the self-coincidence problem.

0. Introduction

Nielsen fixed point theory has been extended for fiber-preserving maps mainly in two different directions. One extension is obtained by considering fiber-preserving maps $f : E \to E$ where $p : E \to B$ is a bundle. Recall that f is a fiber-preserving map if $p \circ f = p \circ \bar{f}$ for some map $\bar{f} : B \to B$, which we call the induced map on the base B. Then we study the fixed points by looking at all deformations of f by a homotopy which at each level is a fiber-preserving map. Several works in the past 40 years have considered this problem. To exemplify two of the pioneer works of this study, see R.F. Brown [2,3].

The other extension is obtained by considering fiber-preserving maps $f : E \to E$ where $p : E \to B$ is a bundle but under the restriction that the induced map $\bar{f}$ on the base is the identity. Then we consider homotopies of f that at each level is a fiber-preserving map that induces the identity on the base. Then one studies the fixed points in this context. Two pioneer works for this study was done by A. Dold [4] and E. Fadell, S. Husseini [5].

The study of coincidence theory of a pair of maps also extends to fiber-preserving maps $f, g : E_1 \to E_2$, where $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ are bundles, and in two different directions, just as for fixed-point theory. In this paper, we study the coincidence theory of fiber-preserving maps of the second type referred to above. More precisely, consider maps $f, g : E_1 \to E_2$, where $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ are bundles over the same space B, and $p_2 \circ f = p_1$, $p_2 \circ g = p_1$. We only consider homotopies H of the maps such that the homotopy at each level H_t is a map which commutes with the projection, i.e. $p_2 \circ H_t = p_1$. Such maps and homotopies are called *maps and homotopies over B*, respectively. Several particular cases of this coincidence problem have been considered for many years by several authors. For example,

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a) If B is a point this question reduces to the usual coincidence theory.
b) If f is the identity and we consider only deformations of g, this is the fixed-point problem over B. See, for example, [4,5,7,9–11].
c) As in case b), for the particular case where the base is S^1, see [9,10,7].
d) For S^1-bundles over S^1, see [8].

Here, in more detail, we study the question stated above in the case when the fiber bundles have the same fiber, and this fiber is the n-dimensional sphere S^n. Let $E \to B$ denote an S^n-bundle over B. We first prove the existence of a fiber-preserving map which is fixed-point free. Then we solve the coincidence problem. For the case of fiber bundles over S^1, we describe the homotopy classes of maps over S^1, and we state the results in terms of this description. Finally we discuss the self-coincidence case.

Let $\tau : E_2 \to E_2$ be a fiberwise fixed point free map (see Proposition 2.1). We show:

Theorem 2.3. *Let $f : E_1 \to E_2$ be a fiber-preserving map. Then there is an unique homotopy class of maps given by $\mathcal{C}(f) = [\tau \circ f]$ such that the pair (f, g) for $g \in \mathcal{C}(f)$ can be deformed over B to a coincidence free pair. Furthermore, if the two bundles are induced by different homomorphisms of $\pi_1(B)$ on $\pi_1(B\mathcal{G}_n)$ then a fiber-preserving map restricted to the fiber has degree zero.*

For S^n-bundles over S^1 we classify all homotopy classes of pairs (f, g) of fiberwise maps between two such bundles. To exemplify let $n = 2$ and $M(A)$ be the twisted bundle over S^1.

Proposition 3.2.

1) *There exists a bijective map*

$$\left[S^2 \times S^1, S^2 \times S^1\right]_{S^1} \to \coprod_{k\in\mathbb{Z}} \left(\pi_1\left(S^{2^{S^2}}; f_k\right) \approx \mathbb{Z}_{2k}\right).$$

2) *There exists a bijective map*

$$\left[M(A), M(A)\right]_{S^1} \to \coprod_{k\in\mathbb{Z}} \left(\pi_1\left(S^{2^{S^2}}; f_k\right)/2\pi_1\left(S^{2^{S^2}}; f_k\right) \approx \mathbb{Z}_2\right).$$

3) *There exists a bijective map*

$$\left[M(A), S^2 \times S^1\right]_{S^1} \to \pi_1\left(S^{2^{S^2}}; f_0\right)/2\pi_1\left(S^{2^{S^2}}; f_0\right) \approx \mathbb{Z}_2$$

where f_0 is a constant map.

4) *There exists a bijective map*

$$\left[S^2 \times S^1, M(A)\right]_{S^1} \to \pi_1\left(S^{2^{S^2}}; f_0\right) \approx \mathbb{Z}$$

where f_0 is a constant map.

Theorem 3.3. *In case* 1), *for each map f over S^1 there is an unique homotopy class of maps* $[g]$ *over S^1 such that* $\deg g_| = -\deg f_|$, $[g] = [\mathcal{A} \circ f]$ *and the pair (f, g) can be made coincidence free. For* $\deg f_| = 0$ *we have that* $[g] = [f]$.

In case 2), *for each map f over S^1 there is an unique homotopy class of map* $[g]$ *over S^1 such that* $\deg g_| = -\deg f_|$, $[g] = [\mathcal{A} \circ f]$ *and the pair (f, g) can be made coincidence free. For* $\deg f_| = 0$ *we have that* $[g] = [f]$.

In cases 3) *and* 4) *a map over S^1 will have the property that restricted to the fiber it has degree zero. In case* 3) *the set of homotopy classes of maps over S^1 is in one-to-one correspondence with $\mathbb{Z}_2$, and in case* 4) *the set of homotopy classes of maps over S^1 is in one-to-one correspondence with $\pi_1(S^{2^{S^2}}; f_0) \cong \mathbb{Z}$, where f_0 is a constant map. Moreover the pair (f, g) can be deformed to be coincidence free if and only if $[f] = [g]$.*

In general, for a pair of maps (f, g) which cannot be made coincidence free, an interesting question is to ask for properties of the coincidence set. Even for the cases of fiber-preserving maps between S^n-bundles over S^1, the answer is not clear.

The following proposition, which is useful to obtain the results of Section 2 and to study the question above, seems to us that it has interest in its own right.

Proposition 1.2. *Let $f, g : E_1 \to E_2$ be a pair of maps over B where the fiber, the base and the total space of the second fiber bundle are closed manifolds. Let (f', g') be a deformation over B of (f, g) and $C = coin(f', g')$. Then there is a map g'' which is a deformation of g over B such that $coin(f, g'') \subset C$. In particular if the pair (f, g) can be deformed over B to (f', g') coincidence free then there is a map g'' which is a deformation of g over B such that (f, g'') is coincidence free.*

This paper is divided into five sections besides this one and Appendix A. In Section 1 we show some results about fiberwise maps which will be used. In Section 2 the main result is Theorem 2.3 and it is proved. In Section 3 we study maps between S^2-bundles over S^1. The main result is Theorem 3.3. In Section 4 we study maps between S^n-bundles over S^1 for $n \geqslant 3$. In Section 5 we discuss the self-coincidence problem for S^n-bundles not necessarily over S^1.

1. Preliminaries

In this section we recall the classification of S^n-bundles over a space B and derive some results about coincidence of fiberwise maps over B.

Following Stasheff [15] we have that spherical bundles over a space B are classified by homotopy classes of maps $[B, B\mathcal{G}_n]$ where $B\mathcal{G}_n$ is the classifying space of the group $\mathcal{G}_n$ of the self-homotopy equivalences of S^n. One can divide the set $[B, B\mathcal{G}_n]$ as follows. Recall that $\pi_1(B\mathcal{G}_n)$ is abelian and isomorphic to $\mathbb{Z}_2$. So we have a map $[B, B\mathcal{G}_n] \to Hom(\pi_1(B), \mathbb{Z}_2)$ for some choice of base points. For each $\theta \in Hom(\pi_1(B); \mathbb{Z}_2)$ consider the subset $[B, B\mathcal{G}_n]_\theta \subset [B, B\mathcal{G}_n]$ of the classes such that the induced homomorphism, by the elements of the class on the fundamental group is θ. For θ trivial these classes correspond to the orientable bundles. We will consider $n \geqslant 2$ and for the case $n = 1$ see [7,8]. In the case $n = 2$ from [13] we have that $\mathcal{G}_2$ has the homotopy type of the product $SO(3) \times \tilde{\Omega}_o^2(S^2)$.

In case of spherical bundles of vector bundles a similar and simpler analysis can be made by replacing $\mathcal{G}_n$ by $O(n+1)$.

In general is not clear how to describe the set of fiber-preserving maps $E_1 \to E_2$ for two S^n-bundles over B. Nevertheless the following lemma is helpful.

Lemma 1.1. *Let E_i be a S^n-bundle, which is classified by a class in $[B, B\mathcal{G}_n]_{\theta_i} \subset [B, B\mathcal{G}_n]$, for $i = 1, 2$. If $f : E_1 \to E_2$ is a map over B and $\theta_1 \neq \theta_2$, then f restricted to the fiber is null homotopic.*

Proof. Because $\theta_1 \neq \theta_2$ it follows that there exists an element $\alpha \in \pi_1(B)$ such that $\theta_1(\alpha) \neq \theta_2(\alpha)$. So either $\theta_1(\alpha) = 0$ and $\theta_2(\alpha) = 1$, or $\theta_1(\alpha) = 1$ and $\theta_2(\alpha) = 0$. Let $A_i : S^n \to S^n$ be a map where $\deg A_i = 1$ if $\theta_i(\alpha) = 0$ and $\deg A_i = -1$ if $\theta_i(\alpha) = 1$. In case $\theta_1(\alpha) = 0$ and $\theta_2(\alpha) = 1$, from the naturality of the action of the $\pi_1(B)$ on $[S^n, S^n]$, given by $\alpha * [g] = [\theta_i(\alpha) \star g]$ where

$$\theta_i(\alpha) \star g = \begin{cases} g & \text{if } \theta_i(\alpha) = 0, \\ A_i \circ g & \text{if } \theta_i(\alpha) = 1, \end{cases}$$

we obtain that the map $f_|$ (the restriction of f to the fiber) has the property that $f_| \circ g \simeq A_2 \circ f_| \circ g$ for all $g : S^n \to S^n$. In particular when $g = I_{S^n}$ we have $f_| \simeq A_2 \circ f_|$. In case $\theta_1(\alpha) = 1$ and $\theta_2(\alpha) = 0$ we obtain $f_| \simeq f_| \circ A_1$. In either case we have that $f_|$ is null homotopic. □

In classical coincidence theory, R. Brooks in [1] has shown that the coincidence theory of maps into a manifold (without boundary) can be obtained by deforming only one of the maps. This fact has been shown to be useful. A weak version of Brook's result also holds for maps over B.

Let us assume that $F_2 \to E_2 \to B$ is a bundle where all the spaces are manifolds. Then we have the following result for maps over B.

Proposition 1.2. *Let $f, g : E_1 \to E_2$ be a pair of maps over B where the fiber, the base and the total space of the second fiber bundle are closed manifolds. Let (f', g') be a deformation over B of (f, g) and $C = coin(f', g')$. Then there is a map g'' which is a deformation of g over B such that $coin(f, g'') \subset C$. In particular if the pair (f, g) can be deformed over B to (f', g') coincidence free then there is a map g'' which is a deformation of g over B such that (f, g'') is coincidence free.*

Proof. From Proposition 2.1 in [5] we have $(E_2 \times_B E_2, E_2 \times_B E_2 - \Delta, \pi_1, E_2)$ is a locally trivial fibered pair. Then there exists (F_t, G_t) such that the following diagram is commutative

$$\begin{tikzcd}
\big(E_1 \times 0, (E_1 - C) \times 0\big) \arrow[r, "{(f',g')}"] \arrow[d] & (E_2 \times_B E_2, E_2 \times_B E_2 - \Delta) \arrow[d, "\pi_1"] \\
\big(E_1 \times I, (E_1 - C) \times I\big) \arrow[r, "{h_t(f',f)}"'] \arrow[ur, "{(F_t,G_t)}"] & E_2,
\end{tikzcd}$$

where $h_t(f', f)$ is a homotopy beginning at f' and ending on f. Let $g'' = G_1$ and we have $(F_1, G_1) = (f, g'')$. It is easy to verify that $coin(f, g'') \subset C$. The particular case follows immediately from the above. □

Now we will show how to interpret homotopy classes of fiberwise maps between two S-bundles over S^1 in terms of homotopy classes of path in certain function spaces, where S is a path connected space. We denote $I = [0, 1]$, $h_j : S \to S$

for $j=1,2$ two homeomorphisms of S and we consider the quotient map $q_j : S\times I \to \frac{S\times I}{(x,0)\sim(h_j(x),1)}$, $j=1,2$. The quotient space $\frac{S\times I}{(x,0)\sim(h_j(x),1)}$ is denoted by E_j, $j=1,2$.

Given $f : E_1 \to E_2$ there is an unique map $F : S\times I \to S\times I$ which covers f and commutes with the projections q_1, q_2.

The restrictions of F to $S\times\{0\}$, $S\times\{1\}$, denoted by $\bar f_0$, $\bar f_1$, respectively, they satisfy the following relation: $\bar f_1 = h_2\circ \bar f_0 \circ h_1^{-1}$. Given $f : E_1 \to E_2$ denote by $\alpha(f)\in (S^S)^I$ the path defined by the adjoint of the first coordinate map of F.

If $\theta : I \to S^S$ is a path denote by $\underline{h_2}\theta\overline{h_1^{-1}} : I \to S^S$ the path given by $\underline{h_2}\theta\overline{h_1^{-1}}(t) = h_2\circ\theta(t)\circ h_1^{-1}$.

Proposition 1.3. *Let $f, g : E_1 \to E_2$ be two fiber maps such that $\bar f_0 = \bar g_0$. Then these two maps are fiberwise homotopic if and only if there is a loop θ at $\bar f_0$ such that the paths $\alpha(f)$ and $\theta * \alpha(g) * (\underline{h_2}\theta\overline{h_1^{-1}})^{-1}$ are homotopic relative to the end points.*

Proof. The "if part". Given a loop θ at $\bar f_0$ and a homotopy $G : I\times I \to S^S$ such that $G(t,0)=\alpha(f)(t)$, $G(t,1) = \theta*\alpha(g)*(\underline{h_2}\theta\overline{h_1^{-1}})^{-1}(t)$, $G(0,s)=\bar f_0$ and $G(1,s)=\bar f_1$, then G defines a homotopy $H : S\times I\times I \to S\times I$ given by

$$H(x,t,s) = \begin{cases} (G(t,2s)(x),t) & \text{if } 0\leqslant s\leqslant 1/2,\\ (G(t+(2s-1)(1-2t)/3,1)(x),t) & \text{if } 1/2\leqslant s\leqslant 1\end{cases}$$

such that $H(x,t,0)=F(x,t)$, $H(x,t,1)=G(x,t)$, $H(x,0,s)=((\bar f_0*\theta)(s)(x),0)$ and $H(x,1,s)=((\bar f_1*(\underline{h_2}\theta\overline{h_1^{-1}}))(s)(x),1)$.

This induces a fiberwise homotopy $T : E_1\times I \to E_2$ between f and g given by $T(\langle x,t\rangle,s) = \langle H(x,t,s)\rangle$.

Conversely, given a homotopy over S^1 between f and g then this homotopy lifts to a homotopy $H : S\times I\times I \to S\times I$ given by $H(x,t,s) = (H_1(x,t,s),t)$ such that $H_1(x,t,0)=\alpha(f)(t)(x)$ and $H_1(x,t,1)=\alpha(g)(t)(x)$. Then $\theta : I\to S^S$ given by $\theta(s)(x) = H_1(x,0,s)$ defines a loop in $\bar f_0$ because $\bar g_0 = \bar f_0$ and $H_1(x,1,s) = (\underline{h_2}\theta\overline{h_1^{-1}})(s)(x)$ because $H_1(h_1^{-1}(x),0,s) = h_2^{-1}H_1(x,1,s)$. Therefore $G : I\times I \to S^S$ given by

$$G(s,u)(x) = \begin{cases} \theta(3s)(x) & \text{if } 0\leqslant s\leqslant u/3,\\ H_1(x,(3s-u)/(3-2u),u) & \text{if } u/3\leqslant s\leqslant 1-u/3,\\ (\underline{h_2}\theta\overline{h_1^{-1}})^{-1}(3s-2)(x) & \text{if } 1-u/3\leqslant s\leqslant 1\end{cases}$$

satisfies $G(s,0)(x)=\alpha(f)(s)(x)$, $G(s,1)(x) = (\theta*\alpha(g)*(\underline{h_2}\theta\overline{h_1^{-1}})^{-1})(s)(x)$, $G(0,u)(x)=\bar f_0(x)$ and $G(1,u)(x)=\bar f_1(x)$. □

We know that given two points $x_0, x_1\in X$ of the same path connected component of X and λ a path from x_0 to x_1, then by means of λ we can identify the homotopy classes of the paths connecting x_0 to x_1 with elements of $\pi_1(X,x_0)$ and also $\pi_1(X,x_1)$ with $\pi_1(X,x_0)$ through an isomorphism. Also given a function $f : X\to X$ such that $f(x_0)=x_1$ then using λ we obtain a self-homomorphism of $\pi_1(X,x_0)$ which is the composite of $f_\#$ with the transport along λ. Consequently we can talk about the Reidemeister classes of this homomorphism which we call *Reidemeister classes of f with respect to λ*. So we can state:

Corollary 1.4. *We can identify the homotopy classes of the paths connecting $\bar f_0$ to $\bar f_1$ with elements of $\pi_1(S^S, \bar f_0)$ by means of some path λ from $\bar f_0$ to $\bar f_1$. The homotopy classes of fiber maps are in one-to-one correspondence with the Reidemeister classes of the map $f\to h_2\circ f\circ h_1^{-1}$ with respect to the path λ. In the particular case where $\pi_1(S^S,\bar f_0)$ is abelian, it follows that this set is in one-to-one correspondence with the quotient of $\pi_1(S^S,\bar f_0)$ by the transport of the subgroup generated by the elements of the form $\theta(\underline{h_2}\theta\overline{h_1^{-1}})^{-1}$ for $\theta\in\pi_1(S^S,\bar f_0)$. In particular if $\pi_1(S^S,\bar f_0)=\mathbb{Z}_2$ then this quotient group is $\mathbb{Z}_2$.*

Proof. The first part is classical. For the second part it follows from Proposition 1.3 that there exists $\theta\in\pi_1(S^S,\bar f_0)$ such that $[\alpha(f)*(\underline{h_2}\theta\overline{h_1^{-1}})] = [\theta*\alpha(g)]$. Therefore $[\alpha(f)*\lambda^{-1}*\lambda*(\underline{h_2}\theta\overline{h_1^{-1}})*\lambda^{-1}] = [\theta*\alpha(g)*\lambda^{-1}]$ and so $\alpha(f)*\lambda^{-1}$ and $\alpha(g)*\lambda^{-1}$ are in the same Reidemeister classes. If $\pi_1(S^S,\bar f_0)$ is abelian then it follows from [6, 2, Example 1] that the set of the Reidemeister classes of the map $f\to h_2\circ f\circ h_1^{-1}$ with respect to the path λ is in one-to-one correspondence with the quotient of $\pi_1(S^S,\bar f_0)$ by the subgroup generated by the elements of the form $\theta*\lambda*(\underline{h_2}\theta\overline{h_1^{-1}})^{-1}*\lambda^{-1}$ for $\theta\in\pi_1(S^S,\bar f_0)$. The rest is straightforward where we use that the only automorphism of $\mathbb{Z}_2$ is the identity. So the subgroup such that the Reidemeister classes are the cosets, is the trivial subgroup and the result follows. □

2. Main results for spherical bundles

Let $E\to B$ be a S^n-bundle.

Proposition 2.1. *There is a fiber-preserving map $\tau : E\to E$ which is fixed point free and the degree of $\tau_{|S^n}$ is 1 if n is odd and it is -1 if n is even. Furthermore, any two such maps are fiberwise homotopic by a homotopy which is fixed point free at each level.*

Proof. Let us consider the fibration $E \times_B E - \Delta \to E$ where the map is the projection on the first variable and the fiber is $S^n - \{y_0\}$. By obstruction theory, see [16, VI, 6, Theorem 6.3], and the fact that the fiber is contractible, it follows that the fibration admits a section. But a section $s : E \to E \times_B E - \Delta$ determines a map $\tau : E \to E$ (the second coordinate) which is fiberwise and has no fixed point since $s(x) = (x, \tau(x)) \in E \times_B E - \Delta$. Since the restriction of τ to the fiber S^n is fixed point free the result about the degree of $\tau_{|S^n}$ follows immediately. This concludes the first part.

For the second part suppose we have two such maps τ_0, τ_1. Then we have two sections $s_i(x) = (x, \tau_i(x))$, $i = 0, 1$. But the obstructions to deform one section into another vertically are trivial [16, VI, 6, Theorem 6.5]. So the two sections are homotopic. So we have $H : E \times I \to E \times_B E - \Delta$ which satisfies $H(x,t) = (x, \tau_t(x))$. The second coordinate provides a homotopy between τ_0 and τ_1. □

Proposition 2.2. *A map τ given by the proposition above has the property that τ^2 is fiberwise homotopic to the identity.*

Proof. The pairs (τ, id) and (τ, τ^2) are coincidence free. So they can be regarded as maps $E \to E \times_B E - \Delta$ and they project (the first coordinate) to the same map. So by similar argument as in the previous proposition the result follows. □

The map τ is like a fixed point free involution up to homotopy.

Remarks 2.1. The map τ is not homotopic over B to the identity if n is even, since its restriction to the fiber S^n has degree -1. For the case where n is odd, despite the fact that the restriction of τ to the fiber is homotopic to the identity, we can have τ not homotopic over B to the identity. The latter case happens when the bundle is $K \to S^1$ where K is the Klein bottle and the fiber map is the usual projection, see [8].

Now we derive the main result for coincidence.

Theorem 2.3. *Let $f : E_1 \to E_2$ be a fiber-preserving map, where $E_1 \to B$ and $E_2 \to B$ are S^n-bundles. Then there is an unique homotopy class of maps given by $\mathcal{C}(f) = [\tau \circ f]$ such that the pair (f, g) for $g \in \mathcal{C}(f)$ can be deformed over B to a coincidence free pair. Furthermore, if the two bundles are induced by different homomorphisms of $\pi_1(B)$ on $\pi_1(B\mathcal{G}_n)$ then a fiber-preserving map restricted to the fiber has degree zero.*

Proof. Let τ be a map given by the previous result, i.e. a fiber-preserving map $\tau : E_2 \to E_2$ which is fixed point free. So the pair $(f, \tau \circ f)$ is certainly coincidence free. This show the existence.

By Proposition 1.2, without loss of generality let $(f, g), (f, g')$ be two pairs of coincidence free maps. Then we have a pair of maps $(f, g), (f, g') : E_1 \to E_2 \times_B E_2 - \Delta$. Again by obstruction theory, see Proposition 2.1, we have that these two maps are fiberwise homotopic. This implies that the two maps g, g' are fiberwise homotopic and the first part follows. The second part follows from Lemma 1.1. □

3. S^2-bundles over S^1

In this section we study the coincidence of fiber maps between S^2-bundles over S^1. We describe the homotopy classes of maps between two such bundles and using this description we describe the pairs of maps which can be made coincidence free.

We will use Proposition 1.3. In our case the maps h_1, h_2 are either the identity or a map of degree -1. For convenience of some specific calculation we can choose the map of degree -1 as the antipodal map or the suspension of the conjugation map $r : S^1 \to S^1$. Denote in either case the map of degree -1 by A_1. Recall that there are two non-isomorphic S^2-bundles over S^1. In both cases we have a map $\mathcal{A} : E \to E$ over S^1 that restricted to each fiber is the antipodal map. One of the bundles is the product $S^2 \times S^1$ and the other one, denoted by $M(A)$, is the quotient $S^2 \times I/(x, 0) \sim (A(x), 1)$ with the obvious projection. We will describe the set of homotopy classes over S^1 in the following cases: case 1) $[S^2 \times S^1, S^2 \times S^1]_{S^1}$, case 2) $[M(A), M(A)]_{S^1}$, case 3) $[M(A), S^2 \times S^1]_{S^1}$ and case 4) $[S^2 \times S^1, M(A)]_{S^1}$. For each integer $k \geqslant 0$ let us fix one model of a map, denoted by f_k, which has degree k. Also let us choose f_0 to be a constant map and f_1 the identity map. For $k < 0$ let f_k be the composite $A \circ f_{-k}$ where A is the antipodal map.

Let $h : S^3 \to S^2$ denote the Hopf map and $\underline{A} : S^{2^{S^2}} \to S^{2^{S^2}}$ the map given by $\underline{A}(g) = A \circ g$, for all $g \in S^{2^{S^2}}$.

Lemma 3.1. *We have that $\pi_1((S^2)^{S^2}; f_0) \cong \mathbb{Z}$ where a generator is $[H]$ for $H : S^1 \to (S^2)^{S^2}$ given by $H(x)(y) = h \circ \pi(x, y)$, $x \in S^1$, $y \in S^2$ and π is the projection $S^1 \times S^2 \to \frac{S^1 \times S^2}{S^1 \vee S^2} = S^3$. Furthermore $[\underline{A} \circ H] = [H]$.*

Proof. Let $x_0 \in S^2$ be a base point, $f_0 : S^2 \to S^2$ given by $f_0(y) = x_0$, $\forall y \in S^2$ and we consider the evaluation map $ev : (S^2)^{S^2} \to S^2$ given by $ev(f) = f(x_0)$. Then ev is a fibration with fiber over x_0 given by $(S^2, x_0)^{(S^2, x_0)}$. But ev has a section $s : S^2 \to (S^2)^{S^2}$ given by $s(x) = \bar{x}$, $\forall x \in S^2$ where $\bar{x} : S^2 \to S^2$ is given by $\bar{x}(y) = x$, $\forall y \in S^2$. Then using the homotopy exact sequence associated to the fibration *ev* we obtain $\pi_1((S^2)^{S^2}, f_0) \cong \pi_1((S^2, x_0)^{(S^2,x_0)}, f_0)$. Since $\pi_1((S^2, x_0)^{(S^2,x_0)}, f_0) \cong \mathbb{Z}$ it follows that $\pi_1((S^2)^{S^2}; f_0) \cong \mathbb{Z}$.

A correspondence between $\mathbb{Z}$ and $\pi_1((S^2, x_0)^{(S^2,x_0)}, f_0)$ is given as follows: the element $1 \in \mathbb{Z}$ corresponds to the class of the map $H : S^1 \to (S^2, x_0)^{(S^2,x_0)}$ given by the composite of the Hopf map $h : S^3 \to S^2$ with the projection $\pi : S^1 \times S^2 \to \frac{S^1 \times S^2}{S^1 \vee S^2} = S^3$. The composition with the antipodal A on the left corresponds to the composite $\underline{A} \circ H : S^1 \to (S^2_{x_0})^{S^2_{x_0}}$. But we know that $\underline{A} \circ H = (\deg(A))^2 H = H$ and therefore $[\underline{A} \circ H] = [H]$ and we have that the involution on $\pi_1((S^2_{x_0})^{S^2_{x_0}}, f_0)$ is the identity. $\square$

Remarks 3.1. An immediately consequence of the lemma above is that a pair (f, g) such that $\deg f_| = \deg g_| = 0$ can be deformed to be coincidence free if and only if the two maps are homotopic.

Proposition 3.2. *In case* 1) *there exists a bijective map*

$$\left[S^2 \times S^1, S^2 \times S^1\right]_{S^1} \to \bigsqcup_{k\in\mathbb{Z}} \left(\pi_1(S^{2^{S^2}}; f_k) \approx \mathbb{Z}_{2k}\right).$$

In case 2) *there exists a bijective map*

$$\left[M(A), M(A)\right]_{S^1} \to \bigsqcup_{k\in\mathbb{Z}} \left(\pi_1\left(S^{2^{S^2}}; f_k\right)/2\pi_1\left(S^{2^{S^2}}; f_k\right) \approx \mathbb{Z}_2\right).$$

In case 3) *there exists a bijective map*

$$\left[M(A), S^2 \times S^1\right]_{S^1} \to \pi_1\left(S^{2^{S^2}}; f_0\right)/2\pi_1\left(S^{2^{S^2}}; f_0\right) \approx \mathbb{Z}_2$$

where f_0 is a constant map.

In case 4) *there exists a bijective map*

$$\left[S^2 \times S^1, M(A)\right]_{S^1} \to \pi_1\left(S^{2^{S^2}}; f_0\right) \approx \mathbb{Z}$$

where f_0 is a constant map.

Proof. First we recall from [13] that $\pi_1(S^{2^{S^2}}; f_k)$ is isomorphic to the cyclic group $\mathbb{Z}_{2k}$. Now we apply Corollary 1.4. It suffices to compute the subgroup such that the quotient gives the Reidemeister classes. From the lemma above the result does not depend on h_2. On the other hand if h_1 is the identity we obtain that the subgroup is the trivial subgroup and for h_1 of degree -1 the subgroup is 2 times the group. So the result follows for each case. $\square$

Theorem 3.3. *In case* 1), *for each map f over S^1 there is an unique homotopy class of map $[g]$ over S^1 such that* $\deg g_| = -\deg f_|$, $[g] = [\mathcal{A} \circ f]$ *and the pair (f, g) can be made coincidence free. For* $\deg f_| = 0$ *we have that* $[g] = [f]$.

In case 2), *for each map f over S^1 there is an unique homotopy class of map $[g]$ over S^1 such that* $\deg g_| = -\deg f_|$, $[g] = [\mathcal{A} \circ f]$ *and the pair (f, g) can be made coincidence free. For* $\deg f_| = 0$ *we have that* $[g] = [f]$.

In cases 3) *and* 4) *a map over S^1 will have the property that restricted to the fiber has degree zero. In case* 3) *the set of homotopy classes of maps over S^1 is in one-to-one correspondence with $\mathbb{Z}_2$, and in case* 4) *the set of homotopy classes of maps over S^1 is in one-to-one correspondence with $\pi_1(S^{2^{S^2}}; f_0) \cong \mathbb{Z}$, where f_0 is a constant map. Moreover the pair (f, g) can be deformed to coincidence free if and only if $[f] = [g]$.*

Proof. The proof follows from Theorem 2.3, Proposition 3.2 and Lemma 3.1. $\square$

4. S^n-bundles over S^1 for $n \geqslant 3$

In this section we study the coincidence of fiber maps between S^n-bundles over S^1 for $n \geqslant 3$. We describe the homotopy classes of maps between two such bundles and using this description we describe the pair of maps which can be made coincidence free. Part of the analysis is similar to the one made in the previous section. Under these circumstance we omit some details of the analyzes.

Recall that there are two non-isomorphic S^n-bundles over S^1. In both cases we have a map $\mathcal{A} : E \to E$ over S^1 that restricted to each fiber is the antipodal map. One of the bundles is the product $S^n \times S^1$ with the projection on the second coordinate, and the other one is the quotient of $S^n \times I/(x, 0) \sim (\phi(x), 1)$ with the obvious projection over S^1, for some homeomorphism of S^n. The homeomorphism ϕ is the antipodal map if n is even and the map of degree -1 which changes the sign of the last coordinates of the sphere if n is odd. Denote this total space by K_n. We will divide the study into four cases. Case 1: we have the domain and the target $S^n \times S^1$. Case 2: we have the domain and the target K_n. Case 3: the domain is K_n and the target is $S^n \times S^1$. Case 4: the domain is $S^n \times S^1$ and the target is K_n. For each integer $k \geqslant 0$ let us fix one model of a map from S^n to S^n, denoted by f_k, which has degree k. Also let us choose f_0 be a constant map and f_1 the identity map.

For $k < 0$ let f_k be the composite $\phi \circ f_{-k}$ where ϕ is defined above. Let C_k denote the set of homotopy classes of maps over S^1 such that the map restricted to a fiber has degree k.

Proposition 4.1. *In case* 1) *there exists a bijective map*

$$\left[S^n \times S^1, S^n \times S^1\right]_{S^1} \to \coprod_{k \in \mathbb{Z}} \left(\pi_1\left({S^n}^{S^n}; f_k\right) \approx \mathbb{Z}_2\right).$$

In case 2) *there exists a bijective map*

$$[K_n, K_n]_{S^1} \to \coprod_{k \in \mathbb{Z}} \left(\pi_1\left({S^n}^{S^n}; f_k\right) \approx \mathbb{Z}_2\right).$$

In case 3) *there exists a bijective map*

$$[K_n, S^n \times S^1]_{S^1} \to \pi_1\left({S^n}^{S^n}; f_0\right) \approx \mathbb{Z}_2$$

where f_0 is a constant map.

In case 4) *there exists a bijective map*

$$\left[S^n \times S^1, K_n\right]_{S^1} \to \pi_1\left({S^n}^{S^n}; f_0\right) \approx \mathbb{Z}_2$$

where f_0 is a constant map.

Proof. By Lemma 4.3 $\pi_1({S^n}^{S^n})$ is isomorphic to $\mathbb{Z}_2$ for any base point. Then the result follows by Corollary 1.4. □

In order to analyze the coincidence of pairs of maps we look at several cases separately. First we have:

Lemma 4.2. *If f is a fiber map such that the degree $f_{|S^n}$ is zero, then (f, f) can be deformed to a coincidence free pair. Therefore (f, g) can be deformed to be coincidence free if and only if $[f] = [g]$.*

Proof. Let f be a fiber map such that the degree $f_{|S^n}$ is zero. Our goal is to show that $[\mathcal{A} \circ f] = [f]$. The map $[f] \to [\mathcal{A} \circ f]$ provides an involution on the set C_0, which is in one-to-one correspondence with $\mathbb{Z}_2$. The bijection between C_0 and $\mathbb{Z}_2$ is given as follows: The $1 \in \mathbb{Z}_2$ corresponds to the composite of the element of the homotopy of the sphere η which is represented by a non-trivial map $S^{n+1} \to S^n$ with the projection $S^1 \times S^n \to S^{n+1}$. The composition with $\mathcal{A}$ corresponds to compose the above map with the antipodal on the left. But we know that $\mathcal{A} \circ \eta = \eta$, since the maps are suspension, see [16]. So the result follows. □

Remarks 4.1. The lemma above is not true for $n = 1$, see Appendix A.

In case 1 given a map f over S^1, this map defines an integer k, which is the degree of f restricted to the fiber, and determines a loop on the function space ${S^n}^{S^n}$ with base point the map $f_{|S^n}$. We can assume that $f_{|S^n}$ is f_k.

Lemma 4.3. *We have $\pi_1({S^n}^{S^n}; f_k) \cong \mathbb{Z}_2$. The map $\underline{A} : {S^n}^{S^n} \to {S^n}^{S^n}$ given by $\underline{A}(f) = A \circ f$ induces $(\underline{A})_\# : \pi_1({S^n}^{S^n}; f_k) \to \pi_1({S^n}^{S^n}; A \circ f_k = f_{(-1)^{n+1}k})$ which is the identity* (*after the suitable identification of paths and loops*) *if and only if either $k = 0$ or n is odd. Therefore a pair (f, f) can be deformed to be coincidence free if and only if either $k = 0$ or n is odd.*

Proof. Let $*$ be a base point in S^n and ${S^n}^{S^n} \xrightarrow{ev} S^n$ the evaluation at $*$. This map is a fibration with fiber over $*$ the function space $(S^n, *)^{(S^n, *)}$. From the long exact sequence of homotopy groups associated to the fibration above we get that $\pi_1({S^n}^{S^n}, f_k) \cong \pi_1((S^n, *)^{(S^n, *)}, f_k)$. But $(S^n, *)^{(S^n, *)} = \Omega^n(S^n)$ and so $\pi_1({S^n}^{S^n}, f_k) \cong [S^1, \Omega^n(S^n)] = [\Sigma^n S^1, S^n] = \Pi_{n+1}(S^n) \approx \mathbb{Z}_2$ [14]. This concludes the first part. For the second part first we observe that the "only if part" is clear since we must have $k = (-1)^{n+1}k$. The "if part" follows from Corollary 1.4 and Theorem 3.3. □

Let n be even.

Lemma 4.4. *In case* 2) *for each map f over B, there is an unique homotopy class of map $[g]$ such that $g \in C_{-k}$ and such that the pair (f, g) can be made coincidence free. Further $[g] = [\mathcal{A} \circ f]$, and $[g] = [f]$ if and only if $k = 0$.*

Proof. That $g \in C_{-k}$ follows because the antipodal has degree -1. The rest follows immediately from the uniqueness and Lemma 4.2. $\square$

It remains to consider n odd, case 2) and degree of $f_{|S^n}$ not zero.

From now on let n be odd. We observe that for n odd the map $[f] \to [\mathcal{A} \circ f]$ is a self-map of C_k which is an involution of C_k. So we would like to decide for which maps f we have $[f] = [\mathcal{A} \circ f]$.

For maps between two S^n-bundles over S^1 we have the following result:

Proposition 4.5. *Let n be odd and $f : K_n \to K_n$ a fiber map. Then (f, f) can't be deformed to be coincidence free if* $\deg(f_{|S^n}) \neq 0$.

Proof. The proof here is a generalization of the proof given in Appendix A for the case $n = 1$. First we are going to show that the identity and antipodal map $\mathcal{A}$ are not in the same homotopy class over S^1 if $n \geqslant 3$ is odd. Let us consider $SO(2n+2) \subset {S^{2n+1}}^{S^{2n+1}}$ and regard S^{2n+1} as the sphere in $\mathbb{C}^{n+1}$. Let $r_\theta : S^1 \to S^1$ be the rotation of angle θ and $R_\theta(z_1, z_2, \ldots, z_n, z_{n+1}) = (r_\theta(z_1), r_\theta(z_2), \ldots, r_\theta(z_n), r_\theta(z_{n+1}))$. It holds that $CR_\theta C(z_1, z_2, \ldots, z_n, z_{n+1}) = (r_\theta(z_1), r_\theta(z_2), \ldots, r_\theta(z_n), r_{(-\theta)}(z_{n+1}))$ where $C : S^{2n+1} \to S^{2n+1}$ is given by $C(z_1, z_2, \ldots, z_n, z_{n+1}) = (z_1, z_2, \ldots, z_n, \overline{z_{n+1}})$ and $\overline{z_{n+1}}$ is the conjugate of z_{n+1}. Let us consider a homotopy H of the constant path $\alpha : I \to {S^{2n+1}}^{S^{2n+1}}$ at the point A such that $H(0) = \alpha$, $H(0,1) = id_{S^{2n+1}}$, $H(1,1) = id_{S^{2n+1}}$, $H(1,t) = CH(0,t)C$. It is not difficult to construct such H which also has the property that $H(s,1) : S^{2n+1} \to S^{2n+1}$ is given by

$$H(s,1)(z_1, z_2, \ldots, z_n, z_{n+1}) = \begin{cases} R_{2\pi s}(z_1, z_2, \ldots, z_n, z_{n+1}) & \text{if } 0 \leqslant s \leqslant 1/2, \\ (r_{2\pi(1-s)} z_1, \ldots, r_{2\pi(1-s)} z_n, r_{2\pi s} z_{n+1}) & \text{if } 1/2 \leqslant s \leqslant 1. \end{cases}$$

This loop is homotopic to the loop given by $\beta(s)(z_1, z_2, \ldots, z_n, z_{n+1}) = (z_1, z_2, \ldots, z_n, r_{2\pi s}(z_{n+1}))$. From the inclusion $SO(2) \hookrightarrow SO(2n+2)$ on the last two coordinates follows that the homotopy class of the loop above is non-trivial and the result follows. Now let us consider the general case, i.e. the map restricted to the fiber has degree $k \neq 0$. We work on the space of the functions space of the base points maps $(S^n, *)^{(S^n, *)}$. We use the fact that the generator for $\pi_1((S^n, *)^{(S^n, *)}, f_k)$ is the homotopy class of the loop $\alpha : I \to (S^n, *)^{(S^n, *)}$ such that $\alpha(t)$ is given by the composite of f_k with the rotation of the angle $2\pi t$ around a fixed axis. Now we use the procedure for $n = 1$ to obtain that $\mathcal{A} \circ f$ is not homotopic to f and the result follows. $\square$

Such examples of (f, f) which cannot be deformed to be coincidence free do not exist in the absolute case for maps into a sphere S^n of odd dimension.

5. Self-coincidences

In this section we make a few considerations about the self-coincidence problem where we assume that the two spherical fiber bundles have the same fiber, but the base B is arbitrary. For the case where $B = S^1$ the coincidence free problem has been analyzed in the previous section, in particular the self-coincidence problem.

Given a map $f : E_1 \to E_2$, where E_i are S^n-bundles over B, we would like to know if the pair (f, f) can be deformed to a coincidence free pair. First we derive some simple necessary conditions.

Lemma 5.1. *If a pair (f, f) can be deformed to a coincidence free pair then either n is even and degree of $f|_{S^n}$ is zero, or n is odd.*

Proof. Let $f_{|S^n} : S^n \to S^n$ be the restriction of f to the fiber. Then $(f_{|S^n}, f_{|S^n})$ can be deformed to a coincidence free pair which implies that $f_{|S^n}$ must be homotopic to $A \circ f_{|S^n}$. From known results about self-maps of the sphere the results follows. $\square$

Observe that if E_1, E_2 satisfy the hypothesis of the Lemma 1.1 then the necessary condition $\deg(f_{|S^n}) = 0$ given by the Lemma 5.1 always holds (see Lemma 1.1).

The converse of Lemma 5.1, by Lemma 4.2, holds for S^n-bundles over S^1 for n even, and for $n > 1$ odd and degree of $f|_{S^n}$ zero. From Appendix A it does not hold for S^1-bundles over S^1. More precisely, for any map $f : K \to K$ over S^1, the pair (f, f) cannot be deformed coincidence free. Also from Proposition 4.2 it does not hold for $n > 1$ odd and degree of $f|_{S^n}$ equal to 1.

We close with an example where the converse of Lemma 5.1, does not hold for n even and degree zero.

Example 5.2. Let us give an example to show that the converse of Lemma 5.1, for n even and degree of $f|_{S^n}$ zero, does not hold. Let $E = S^4 \times S^3$ be the total space of the trivial bundle with fiber S^4, and $f : E \to E$ the map which first coordinate map is the composite of the Hopf map $H : S^7 \to S^4$ with the projection $S^4 \times S^3 \to S^7$. Recall that the adjoint of this composite $\bar{f} : S^4 \times S^3 \to S^4$ is a map $S^3 \to (S^4, *)^{(S^4, *)}$ which provides a fiberwise map of the trivial bundles. But the composite of the Hopf map $H : S^7 \to S^4$ with the antipodal map $A : S^4 \to S^4$, by [12], is $A \circ H = H - \nu$, where ν is a generator of the cyclic part of $\pi_7(S^4)$. Therefore $A \circ H$ is not homotopic to H which implies that f is not homotopic to $\mathcal{A} \circ f$. Since $f_{|S^4 \times *}$ is the constant map the result follows.

For E_1, E_2 S^n-bundles over S^1 the result for self-coincidence can be obtained from Sections 3 and 4.

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Appendix A

Let us consider S^1-bundles over S^1. Among other results, the coincidence theory for fiber maps between these bundles has been completely studied in [8]. Also a classification of the set of homotopy classes of fiber maps between two such bundle is obtained. Here we obtain by different methods the classification of homotopy classes of fiber maps and we classify the pair of maps which can be deformed to be coincidence free. One reason to do this is because the approach used here can be applied to certain S^n-bundles for $n > 1$.

We begin with the classification of the fiberwise maps. First we observe that $\pi_1((S^1,*)^{(S^1,*)})$ is trivial for any base point. This follows from the fact that $\pi_2(S^1,*)$ is trivial. Let us consider the fibration ${S^1}^{S^1} \xrightarrow{ev} S^1$, where the projection ev is the evaluation at $*$ and the fiber over $*$ is $(S^1,*)^{(S^1,*)}$. From the homotopy long exact sequence it follows that $\pi_1({S^1}^{S^1})$ is isomorphic to $\pi_1(S^1) \approx \mathbb{Z}$.

A fiber map between two S^1-bundles over S^1 determines either a loop or a path on the space of the free loops ${S^1}^{S^1}$. We use the notation of f_k for a self-map of S^1 of degree k as in Sections 3 and 4. The two fiber bundles in our case are $S^1 \times S^1$ and K (Klein bottle), respectively. The cases from 1 to 4, mentioned below, are as the ones in Sections 3 and 4.

Proposition 5.3. *In case* 1) *there exists a bijective map*

$$\big[S^1 \times S^1, S^1 \times S^1\big]_{S^1} \to \coprod_{k \in \mathbb{Z}} \Big(\pi_1\Big({S^1}^{S^1}; f_k\Big) \approx \mathbb{Z}\Big).$$

In case 2) *there exists a bijective map*

$$[K, K]_{S^1} \to \coprod_{k \in \mathbb{Z}} \Big(\pi_1\Big({S^1}^{S^1}; f_k\Big)/2\pi_1\Big({S^1}^{S^1}; f_k\Big) \approx \mathbb{Z}_2\Big).$$

In case 3) *there exists a bijective map*

$$\big[K, S^1 \times S^1\big]_{S^1} \to \pi_1\Big({S^1}^{S^1}; f_0\Big) \approx \mathbb{Z}$$

where f_0 is a constant map.

In case 4) *there exists a bijective map*

$$\big[S^1 \times S^1, K\big]_{S^1} \to \pi_1\Big({S^1}^{S^1}; f_0\Big)/2\pi_1\Big({S^1}^{S^1}; f_0\Big) \approx \mathbb{Z}_2$$

where f_0 is a constant map.

Proof. Arguing as in Proposition 1.3, every fiber map provides a loop or a path in ${S^1}^{S^1}$ and after some identification a well defined element of a quotient of $\pi_1({S^1}^{S^1})$. From now on the situation here differs from that one in Proposition 1.3 since we cannot identify a homotopy class in free loops with a homotopy class in based loops. By means of the isomorphism $ev_\# : \pi_1({S^1}^{S^1}) \to \pi_1(S^1) \approx \mathbb{Z}$ we obtain a well defined element of a quotient of $\mathbb{Z}$. In case the target is $S^1 \times S^1$ we obtain $\mathbb{Z}$ and in the case the target is K (so the gluing function is the conjugation) then we obtain $\mathbb{Z}_2$. We apply this for the 4 cases above and the result follows. $\square$

Now we classify the pairs of homotopies $([f],[g])$ such that if $f' \in [f]$ and $g' \in [g]$ then the pair (f', g') can be made coincidence free.

Proposition 5.4. *Let $f : E_1 \to E_2$ be a fiber map where E_1, E_2 are S^1-bundles over S^1. Then*:

1) *If $E_2 = S^1 \times S^1$ then (f, g) can be deformed over S^1 to a coincidence free pair if and only if $[f] = [g]$.*

2) *If* $E_1 = S^1 \times S^1$ *and* $E_2 = K$ *then degree of* $f_{|S^1}$ *is zero and* (f, f) *cannot be deformed over* S^1 *to a coincidence free pair.*
3) *If* $E_1 = E_2 = K$ *then* (f, f) *can't be deformed over* S^1 *to a coincidence free pair for any fiber map* f.

Proof. Part 1). Let $A \times id : S^1 \times S^1 \to S^1 \times S^1$ where $A : S^1 \to S^1$ is the antipodal map. Since $A \times id$ is homotopic over S^1 to $id \times id$, follows that (f, f) is homotopic to $(f, \mathcal{A} \circ f)$, which is a coincidence free pair. The "only if part" follows by the uniqueness given in Section 2.

Part 2). We have that the map f restricted to the fiber has degree zero and we also know that we have two homotopy classes. Let us consider the map f_1 which is the composite of the projection followed by the section s_0 of $K \to S^1$ given by $s_0(x) = (x, 0)$ (see [8]). If we compose f_1 with $\mathcal{A}$ we obtain the map f_2 which is the composite of the projection with the section $s_1(x) = (x, 1/2)$. If we deform f_2 such that at the fiber S^1 is the constant at $(0, 0)$ then we get a loop on ${S^1}^{S^1}$ which projects by the evaluation on the generator of $\pi_1(S^1)$ consequently non-trivial in $\mathbb{Z}_2$. So the result follows.

Part 3). Let us suppose that degree of $f_{|S^1}$ is one. Consider the two functions the identity and the antipodal map $\mathcal{A}$. We are going to show that they are not in the same homotopy class over S^1. Let us consider a homotopy H of the constant path $\alpha : I \to {S^1}^{S^1}$ at the point A such that $H(\ , 0) = \alpha$, $H(0, 1) = id_{S^1}$, $H(1, 1) = id_{S^1}$, $H(1, t) = CH(0, t)C$, where $C : S^1 \to S^1$ is the complex conjugation. It is not difficult to construct such H which also has the property that $H(s, 1) : S^1 \to S^1$ is the rotation of angle $2\pi s$. But we know that this loop is not homotopic to the constant loop since by means of the induced homomorphism of the evaluation map (as explained in part 2), is mapped to a non-trivial element of $\pi_1(S^1)$, and the result follows. For the cases where the degree of $f_{|S^1}$ is $k \neq 1$ we will reduce the problem to that where the degree is 1. Let $h : K \times_{S^1} K \to K$ be the map which restricted to $S^1 \times S^1 \subset K \times_{S^1} K$ is defined by $h(x, y) = xy^{-1}$ using the multiplication of S^1. This is a well defined map since it is compatible with the conjugation used to construct K. Then we consider the maps $h \circ (f, f_{k-1})$ and we get the result. □

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